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THE PRINCIPLE OF LOCAL ROTATIONAL INVARIANCE AND THE COEXISTENCE OF MAGNETISM, CHARGE AND SUPERCONDUCTIVITY

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We propose a macroscopic description of the superconducting state in presence of an applied external magnetic field in terms of first order differential equations. They describe a corrugated two-component order parameter intertwined with a spin-charged background, caused by spin correlations and charged dislocations. The first order differential equations are a consequence of a Weitzenböck-Liebnorowitz identity which renders a $SU_L(2) \otimes U_L(1)$ invariant ground state, based on (L) local rotational and electromagnetic gauge symmetry. The proposal is based on a long ago developed formalism by Élie Cartan to investigate curved spaces, viewed as a collection of small Euclidean granules that are translated and rotated with respect to each other. Élie Cartan's formalism unveils the principle of local rotational invariance as a gauge symmetry because the global $SU(2)$ invariance of the order parameter is turned into a local invariance by the interlacement of spin and charge to pairing.

Keywords: Phenomenological theories; two-fluid, Ginzburg-Landau; Superconductivity phase diagrams; Magnetic properties; vortex structures and related phenomena

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1. Introduction

The present experimental evidence indicates a far reaching complexity of the superconducting state in the new compounds not found before in metals. Different electronic orders seem to coexist inside the compound. It is even conceivable that a single set of electrons participate in different orders, or more likely, that nano separated neighbor electrons belong to different orders. The evidence to the coexistence of different orders has been accumulating since the early days of high-temperature ceramic superconductors when physicists stomped into the pseudogap ¹, a gap that sets in at a temperature much above the critical temperature. Recent evidence shows that electrons in the pseudogap phase are not paired up, but organized in a new order that persists when the compound becomes superconductor ². Another remarkable feature of superconductivity in the new compounds is its magnetism ³. Magnetism was once thought to be detrimental to the superconducting electron pairs, now is acknowledged to coexist and possibly contribute to pair stability. Nuclear magnetic resonance studies have revealed that spin correlations coexist with superconductivity in the superconducting cuprates varying from long-range to short-range correlations according to doping, as depicted in Fig.1. Therefore the evidence is that superconductivity is intertwined with spin and charge degrees of freedom ⁴ forming a highly correlated electronic system. The onset of different types of inhomogeneous states with broken rotational and translational symmetries, such as striped, nematic and smectic phases ^{4,5,6} can be understood as a natural consequence of such coexistence. A general theoretical framework able to deal with this plethora of phenomena is still missing. Nevertheless the coexistence of spin, charge, and pairing orders ^{7,8,9,10,11}, the evidence of multigap superconductivity ^{12,13,14,15}, and the layered structure seem to be common features of the new compounds.

In this paper we consider the principle of local rotation as a way to treat all these common features and describe the onset of heterogenous states. The simplest possible framework to describe the superconducting state is through the quantum macroscopic approach, whose goal is to describe the superconducting state by means of an order parameter (OP). Within this framework we study this principle of local rotational invariance taking that the non-superconducting degrees of freedom are spin correlations and charged dislocations interlaced with the superconducting OP to form a heterogeneous state. The superconducting OP feels the presence of the spin correlations and charged dislocations to become spatially corrugated. Reversely the corrugated OP act in these non-superconducting degrees of freedom, but this is not treated here, and for this reason we refer to the presence of these spin correlations and charged dislocations as *the spin-charged background*.

The quantum macroscopic approach is very successful to describe superconducting phenomena and yet it was formulated without knowledge that electrons pairs form the ground state of the superconducting state, which is a key ingredient of the macroscopic theory. In 1950, Vitalii Ginzburg and Lev Landau published their

phenomenological theory of superconductivity by including the principle of gauge invariance into the general theory of the second order phase transitions proposed earlier by Landau in 1937. For this purpose, the OP was set complex in order to have minimal coupling to the magnetic field, a puzzling assumption later proven to be correct. The Meissner effect was explained on this basis, and so, it can be regarded as a natural consequence of the gauge invariance of the Ginzburg-Landau theory. Similarly, the present formalism proposes a new gauge symmetry to explain the observed inhomogeneity of the superconducting state, that is, to describe the spatial corrugations of the OP in presence of the spin-charged background. This new gauge invariance is the principle of local rotational invariance to be described in this paper.

The ultimate goal of the quantum macroscopic approach is to determine the OP, and in this way to describe the superconducting state. Usually this is achieved by firstly proposing a free energy expansion in powers of the OP, such as in the Ginzburg-Landau theory. However there is a simpler framework to determine the OP, which makes no assumption about the condensate energy and only on the kinetic energy. We call it *the ground state condition*. It was firstly noticed by A. A. Abrikosov, in his fundamental treatment of the Ginzburg-Landau theory¹⁶. He obtained the OP using this condition and found that it also provides an exact solution of Ampère's law, because it directly relates the supercurrent to the superconducting density. The ground state condition does determine the OP and its most relevant aspects, such as the vortex lattice and the magnetization of the superconducting state. Thanks to the ground state condition Abrikosov found that the magnetization is proportional to the spatial average of the OP, $\langle |\psi|^2 \rangle$, where ψ is the one-component OP, without invoking the Ginzburg-Landau theory. The ground state condition is expressed by two first order differential equations that many years after were rediscovered by E. Bogomolny¹⁷ in the context of string theory. He showed that they solve exactly the Ginzburg-Landau theory for a special coupling value, $\kappa = 1/\sqrt{2}$. The two equations are given by,

$$D_+ \psi = 0, \quad \text{and}, \quad (1)$$

$$h_3 = H - \frac{h q}{m c} |\psi|^2. \quad (2)$$

A uniaxial symmetry along the applied field direction must be and here it is set along the x_3 axis, such that $\psi(x_1, x_2)$ and $h_3(x_1, x_2)$ ($h_1 = h_2 = 0$), are determined by these equations ($D_+ = D_1 + i D_2$, $\vec{D} \equiv \frac{\hbar}{i} \vec{\nabla} - \frac{q}{c} \vec{A}$, $A_1(x_1, x_2)$, $A_2(x_1, x_2)$). An iterative way to obtain a solution from these nonlinear equations is to firstly solve Eq.(1) for ψ , under the assumption of a constant applied external field H ($A_1 = 0$ and $A_2 = H x_1$). The first equation is just the lowest Landau level condition whose solution is $\psi = \sum_k c_k \exp \left[i k x_1 - \frac{q H}{2 \hbar c} \left(x_2 + \frac{\hbar c k}{q H} \right)^2 \right]$. The set of wavenumbers k and the constants c_k are determined by imposing periodic conditions to the order parameter and fixing the number of vortices within the unit cell area¹⁸. Next one obtains h_3 from Eq.(2) using the previously determined $|\psi|^2$. The procedure can

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be recursively repeated until convergence is achieved. However just the first step is known to provide an excellent description of the full GL free energy solution for ψ and h_3 in the range $0.5H_{c2} \leq H \leq H_{c2}$, as shown by E.H. Brandt¹⁹. Based on the fact that they are independent of the Ginzburg-Landau theory and its applicability is not restricted to a single value of κ , we conclude that the ground state condition lives in a level more fundamental than that of the free energy expansion. Nevertheless it is not a replacement to the free energy expansion since it describes the vortex lattice, but without determining its symmetry, that can only be known through a minimization procedure of the free energy expansion.

Behind the ground state condition is the so-called Weitzenböck-Liechnorowitz identity, that in the present case is given by the following expression:

$$F_k = \int \frac{d^2x}{A} \frac{1}{2m} (|D_1\psi|^2 + |D_2\psi|^2) = \int \frac{d^2x}{A} \left(\frac{1}{2m} |D_+\psi|^2 + \frac{\hbar q}{2mc} h_3 |\psi|^2 \right). \quad (3)$$

This identity provides a twofold description of the kinetic energy density, F_k . The area orthogonal to the field direction is described by A . This twofold formulation of the kinetic energy leads to a twofold formulation of the supercurrent, which follows from the linear term in the vector potential of the kinetic energy,

$$F_k = \int \frac{d^2x}{A} \left(-\frac{1}{c} \vec{J} \cdot \vec{A} + \dots \right). \quad (4)$$

The first formulation of the kinetic energy gives that,

$$J_a = \frac{q}{2m} [\psi^* (D_a\psi) + \psi (D_a\psi)^*], \quad (5)$$

$a = 1$ or 2 , whereas it follows from the second one that,

$$J_1 = \frac{q}{2m} [\psi^* (D_+\psi) + \psi (D_+\psi)^*] + \frac{\hbar q}{2m} \partial_2 |\psi|^2, \quad (6)$$

and

$$J_2 = \frac{q}{2m} [\psi^* (D_+\psi) - \psi (D_+\psi)^*] - \frac{\hbar q}{2m} \partial_1 |\psi|^2. \quad (7)$$

respectively. Once granted the symmetry along the third axis, Ampère's law is simply given by $\partial_1 h_3 = -4\pi J_2/c$ and $\partial_2 h_3 = 4\pi J_1/c$. Then it becomes straightforward to check that the ground state condition solves Ampère's law.

The ground state condition is also useful to find a reliable, though approximate, solution of the other Ginzburg-Landau equation:

$$\frac{1}{2m} (D_1^2 + D_2^2) \psi = \alpha(T) \psi - \frac{\beta}{2} |\psi|^2 \psi, \quad (8)$$

where $\alpha(T) = \alpha_0(T_c - T)$ and $\beta > 0$. Integration of this equation, by firstly multiplying it by ψ^* , and then using periodic boundary conditions, gives that,

$$\int \frac{d^2x}{A} \left[\frac{1}{2m} (|D_1\psi|^2 + |D_2\psi|^2) - \alpha(T) |\psi|^2 + \beta |\psi|^4 \right] = 0. \quad (9)$$

The ground state condition turns this integrated equation into an algebraic equation, by use of the kinetic energy expression, Eq.(3), and the ground state equations, Eqs.(1) and (2):

$$\frac{\hbar q}{2mc} [H - H_{c2}(T)] \langle |\psi|^2 \rangle + \beta \left(1 - \frac{1}{2\kappa^2}\right) \langle |\psi|^4 \rangle = 0. \quad (10)$$

The average value means $\langle \dots \rangle \equiv \int d^2x \{ \dots \} / A$, the upper critical field is $H_{c2}(T) = (2mc/q\hbar)\alpha(T)$ and $\kappa = \sqrt{\beta/2\pi}(mc/\hbar q)$. From this equation one can easily conclude that if $\kappa > 1/\sqrt{2}$ a non-zero ψ solution is only possible if $H < H_{c2}(T)$. Thus we find that it is possible to determine the upper critical line ($H_{c2}(T)$) without explicit calculating the OP, instead, just assuming that the OP satisfies the ground state equations. Hence the ground state condition solves exactly Ampère's law and also gives relevant information about the OP solution of the other Ginzburg-Landau equation.

We stress the intimate connection between the ground state condition and the kinetic and field energies, but not to the condensate energy. This makes the ground state approach independent of the critical temperature, whose value is determined by the condensate energy, not present in our considerations. Therefore the ground state condition applies both below and above the critical temperature value as solely reflects properties of the kinetic and field energies. The ground state condition can also be derived from the Virial theorem of superconductivity^{20,21}.

Interestingly the ground state equations also appear in the microscopic approach. Long ago the magnetic field distribution of pure type-I superconductors with small magnetization was derived from the non-local version of Gorkov's theory²² and the result is that the local field is equal to the applied field H added to the average gap square^{22,23}, as described by Eq.(2).

For all the above reasons we find relevant the derivation of the ground state condition for the new superconductors. As shown in this paper this derivation follows from a principle of local rotational invariance, which is a local gauge symmetry.

2. The ground state condition for the layered superconductors

In the previous section we have described the ground state condition for the traditional superconductors and shown that these equations stem from the kinetic and field energies. The kinetic energy describes how the OP is coupled to the local vector potential through *minimal coupling* in order to be gauge invariant. The steps followed before also apply here to determine the two-component OP, Ψ , and the local magnetic field, h^k , in presence of a spin-charged background. These steps lead to the following equations:

$$\sigma^j(x) D_j \Psi = 0, \quad \text{and}, \quad (11)$$

$$h^k = H^k - \frac{\hbar q}{mc} \Psi^\dagger \sigma^k(x) \Psi. \quad (12)$$

Notice that these equations also do not include the critical temperature. The spin and charge degrees of freedom of the background enter the equations through the

local Pauli matrices, $\sigma_j(x)$, and the *spin connection* field in the covariant derivative D_j . This covariant derivative is different from the previous one because besides the local magnetic potential it also contains a new gauge field to describe the interaction with the spin-charged background. The study of the local spin and covariant derivative operators is done in the following sections.

Eqs.(11) and (12) reflect profound conceptual changes on the macroscopic approach of superconductivity as compared to Eqs.(1) and (2). They describe the interaction between the OP and the local magnetic field over a spin-charged background defined on a curved space with torsion. Long ago Élie Cartan²⁴ developed the formalism of a curved space with torsion and here we show that it provides the appropriate venue to include spin correlations and charged dislocations.

A few years before the discovery of spin by Uhlenbeck and Goudsmith, Cartan introduced the concept of torsion in general relativity as an intrinsic angular momentum of matter^{25,26,27}, whose importance would be in the same footing as mass. We find that superconductors, which coexist with a spin-charged background, are the true foreground of Cartan's geometrical theory. Élie Cartan's geometrical formalism²⁴ was inspired on an analogy with mechanics of elastic media, like a collection of small granules that are translated and rotated with respect to each other²⁸. We stress a fundamental conceptual difference between the use of Cartan's geometrical formalism to Superconductivity and to General Relativity. In the latter the curvature of space is caused by mass and its torsion by spin, whereas for the former only spin correlations curve the space, as felt by the superconducting carriers. As we shall see here that torsion in our system is caused by charged dislocations. In fact Élie Cartan's geometrical formalism has been applied to solids before. A crystal populated by sufficiently many dislocations can be described by a continuum field theory²⁸, which by its turn is formulated as a gauge theory of dislocations^{29,30,31}. Élie Cartan's formalism³² expresses this gauge theory as three dimensional theory of gravity. A crystal with dislocations and disclinations in Euclidean space can be expressed in curved space with torsion as described by H. Kleinert³³, via a singular coordinate transformation. This description of defects has no local invariance under rotations and therefore is not useful for the present purposes.

Recently dislocations in graphene have been treated as the torsion field using this formalism³⁴. All previous applications of the so-called Einstein-Cartan geometry to solids have been done in the context of crystallographic defects, whereas here we apply this formalism to describe a spin-charged background felt by the superconducting state. A feature of Cartan's geometrical formalism is to turn a curved space theory, like gravity^{35,36,37}, into a Yang-Mills theory^{25,26,38,39}, which is known to describe the fundamental internal symmetries of particle Physics. Nevertheless the internal gauge symmetry of Élie Cartan's approach to gravity is the group of local space-time symmetry.

Notice that Eq.(11) is the Euclidean version of the three-dimensional Dirac

equation, with time replaced by one of the spatial dimensions⁴⁰, treated differently from the other two ones. In the Dirac equation the search for eigenstates breaks the space-time invariance when time is treated differently from the spatial coordinates as it enters the solution through the exponential $\exp(iEt/\hbar)$, where E is the energy eigenvalue to be determined. Similarly, the present Euclidean treatment takes the third coordinate distinctively from the other two, as it enters the solution through the exponential $\exp(-q_3|x_3|)$, where q_3 is an eigenvalue to be determined through the Euclidean Dirac equation turned into an eigenstate equation. According to this view the OP behavior perpendicular to the layer is locked to the behavior along the layer, an important feature for the treatment of multiple layers. This is done through these equations by linear superposition of the individual layers, thanks to the linearity of Eq.(11) in Ψ . The interaction between layers will carry this interdependency between in and out of the layer behavior found here, that may become a fundamental feature to enhance superconductivity, as stressed by some authors^{41,42,43}. The presence of the background and of the applied field perpendicular to the layers yields non-zero solutions for the OP in Eq.(11), and therefore helps to stabilize superconductivity. By its turn Eq.(12) shows that the non-zero OP induces a local magnetic moment along the layers, given by components $\Psi^\dagger \sigma_1(x) \Psi$ and $\Psi^\dagger \sigma_2(x) \Psi$. However the total magnetization summed over all layers must average to zero¹⁸.

We point out to the similarities of the present state with the FFLO state, which exhibits inhomogeneous superconducting phases intercalating spatially oscillating OP and spin polarization^{44,45,46}. This is similar to the present OP corrugated by spin correlations and charged dislocations described by the background. We notice that the ground state condition is a three-dimensional version of the well-known Seiberg-Witten equations^{47,48,49}, originally written for a smooth compact four dimensional manifold.

3. The local momentum and spin operators

The derivation of the ground state equations Eqs.(11) and (12) is set over a kinetic energy expression which has a twofold formulation obtained from a Weitzenböck-Liechnorowitz identity. Therefore we also follow the same pathway for the two-component OP in presence of the spin-charged background. The kinetic energy displays the remarkable property of a *non-abelian gauge symmetry under the group of spatial rotations*. Therefore all results obtained here follow from this principle of local rotational invariance. The mathematical details related to the derivation of the corresponding Weitzenböck-Liechnorowitz identity^{49,50} will be seen elsewhere. This identity strongly relies on the commutativity between the local spin ($\sigma^j(x)$) and the local momentum operators (D_i). These operators carry information about the spin-charged background, and is their locality that render the theory locally invariant under rotations.

Recently a curved space has been used to treat the effect of nematicity in super-

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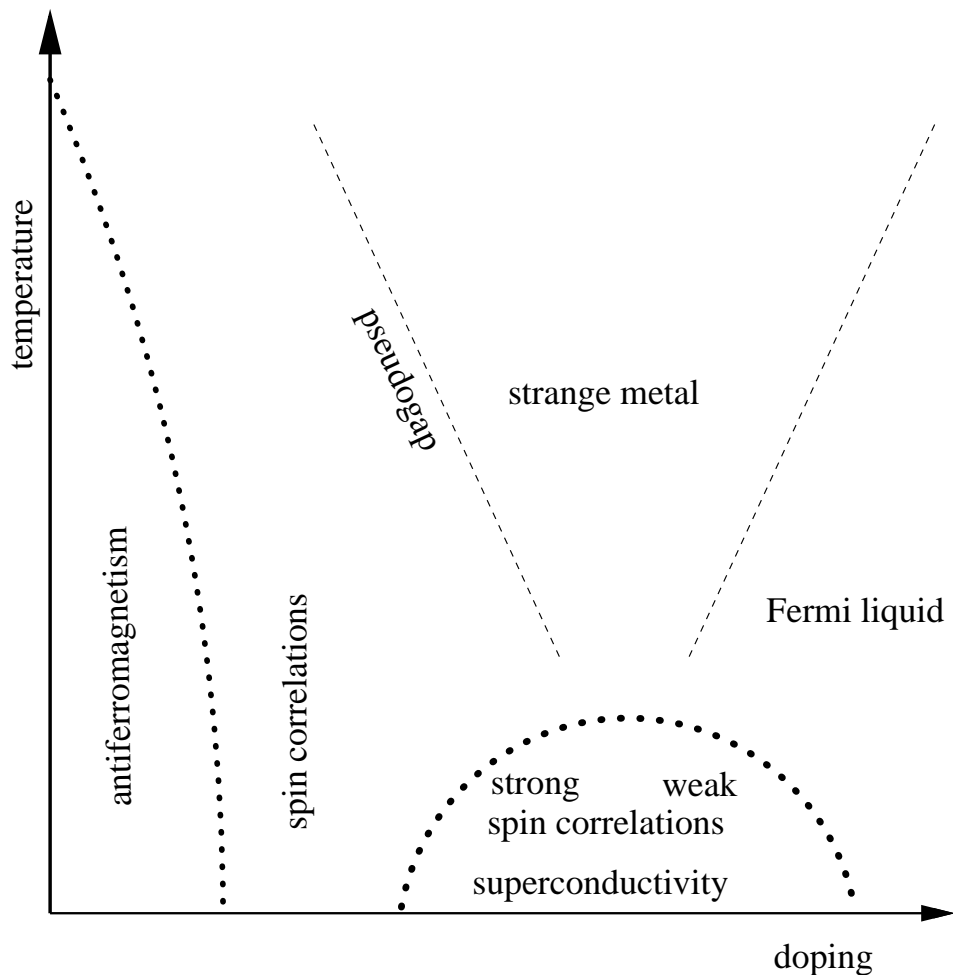


Fig. 1. A schematic phase diagram is shown here for the temperature versus the concentration of charge carriers (electrons or holes), called doping. For low doping the state is of an antiferromagnetic insulator, but carriers become mobile by increasing doping, leading to the pseudogap, the strange metal and the Fermi liquid domains, which are displayed in the phase diagram. Spin correlations are stronger near the antiferromagnetic insulator phase and are also observed within the superconducting dome, although they become weak upon doping.

conductors⁵¹, where nematic order stands for an spontaneously broken rotation symmetry of a lattice system. A suggestive argument to show the relevance of curved spaces in strongly correlated systems stems from Fermi liquid theory. This theory, proposed by Landau in 1956, describes weakly correlated fermions. It was successfully applied to many systems, such as Liquid He-3, electrons in a normal metal and protons and neutrons in an atomic nucleus. A basic concept behind Fermi liquid theory is that of a quasiparticle, which is a particle dressed by its interaction

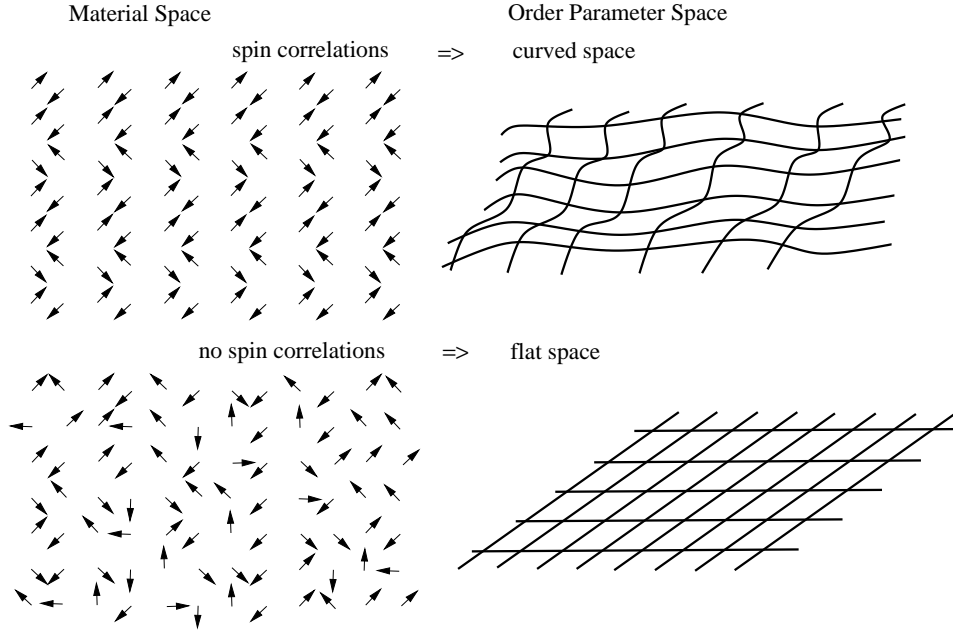


Fig. 2. This is a pictorial view of the differences between the material and the order parameter spaces. The observer's view is in material space where individual charges and spins of static and mobile carriers interact. The order parameter (OP) space is that seen by the superconducting condensate which senses the space curved in regions of intense spin correlations. In regions of uncorrelated spins the space is flat. This follows from the spin-frame, which in uncorrelated regions is a pure rotation, $e_b^i(x) = u_b^i(x)$ and the metric, according to Eq.(18), becomes $g^{ij} = \delta^{ij}$, to describe a flat space.

with the other particles, resulting into a state with characteristics of a free particle, namely, a definite momentum and spin. Both quasiparticles and particles are characterized by momentum, \vec{P} , and spin, $\vec{\sigma}$, and this, obviously, relies on the fact that these two quantities can be simultaneously observed because they are commuting operators:

$$[P_a, \sigma_b] = 0. \quad (13)$$

Assume, for the sake of the argument, that the inhomogeneous spin state of a strongly correlated system turns the spin into a local operator described by $\vec{\sigma}(x)$. The full meaning of this locality is discussed further in the text. The point to make here is that this local dependency breaks the above relation, $[P_i, \sigma^j(x)] \neq 0$, and consequently spoils the concept of a quasiparticle characterized by momentum and spin. The remarkable fact is that it is possible to heal this relation and define a new momentum operator, hereafter called ∇_i able to commute with the local spin operator.

$$[\nabla_i, \sigma^j(x)] = 0. \quad (14)$$

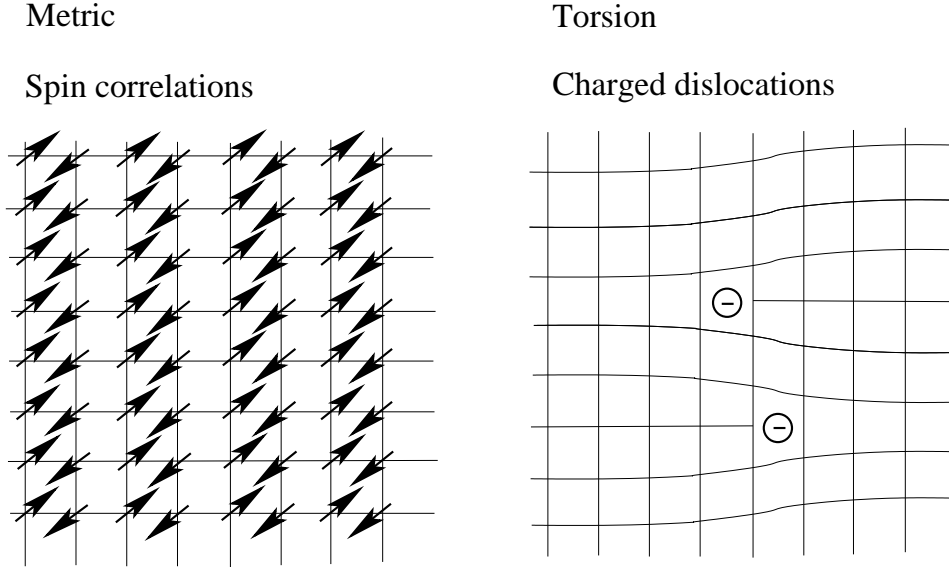


Fig. 3. Spin correlations and charge dislocations are the basic quantities that interact with the order parameter (OP). They are mathematically described by the metric and the torsion. The spin-frame, $e_a^i(x)$ is one of the building blocks of Cartan's geometrical approach, from where the metric follows, $g^{ij} = e_a^i e_a^j$. The torsion is the other building block and corresponds to the antisymmetric part of the affine connection, $\Gamma_{kl}^i \varepsilon^{klm}$.

This commutativity is made possible because space has acquired curvature and torsion by the presence of the spin correlations and charged dislocations, respectively. The above relation is the heart of the present geometrical approach because the momentum operator ∇_i is the so-called *covariant derivative* introduced by Fock and Ivanenko in 1929 in the context of General Relativity⁵² to deal with Dirac spinors. A similar but equivalent condition to the commutativity of Eq.(14) was independently introduced by Élie Cartan²⁴, whose formulation of General Relativity is in terms of the so called "co-frame", a matrix that lives in a more fundamental level than the metric itself. This matrix plays a fundamental role in the present treatment and for this reason we call it "spin-frame", since it contains information about the spin correlations present in the system.

To construct the local commuting momentum and spin operators we firstly digress from superconductivity to discuss the OP of a spin glass, which was shown long ago by many authors to contain more than one spin field^{53,54,55,56,57}. Correlations and not individual spins are important for the spin glass and here we take the same point of view. Thus we define the following quantum correlation between a spin in position x and a reference spin:

$$e_a^i(x) = \langle 0 | \frac{1}{2} \{ \sigma^i(x), \sigma_a \} | 0 \rangle. \quad (15)$$

For the sake of the argument we assume knowledge of the microscopically obtained spin ground state, $|0\rangle$. The relevant fact is that the above equation provides a definition of the spin-frame, $e_a^i(x)$, the building block of Cartan's geometrical approach. Notice that a point x in $e_a^i(x)$ really describes an average value over microscopic spins and therefore refers to a region of degrees of freedom, instead of a single point in space. The microscopic state $|0\rangle$ is fundamental to help us understand how correlations enter the problem, but its derivation is beyond the scope of the present study. For this reason we simply introduce the following definition of the spin-frame, a transformation of the Pauli matrices into a new local set,

$$\sigma^i(x) = e_b^i(x)\sigma^b. \quad (16)$$

This transformation is consistent with the formulation through the quantum correlation given in Eq.(15). Nevertheless it must be kept in mind that this new local set of Pauli matrices does not represent a single spin as the original set does, and truly carries information about local spin correlations that must capture the relevant physical features ranging from a glassy state with frustrated spins to a state of independent uncorrelated spins. Hereafter we assume knowledge of the local 3 by 3 spin-frame matrix $e_a^i(x)$ everywhere. While for a highly correlated spin system the spin-frame is a full matrix, for the uncorrelated case it becomes an orthogonal matrix, $e_a^i(x) = u_a^i(x)$. This is because if all spins are independent rotations of the reference spin, then $\sigma^i(x) = u_b^i(x)\sigma^b$, where $u_b^i(x)$ represents a rotation in space, $u_a^i(x)u_a^j(x) = \delta^{ij}$.

Throughout this paper we employ the Einstein notation, that repeated indices mean a summation. A simple and naive way to describe an intense correlated spin background, is to assume a scaling function, $\lambda^{-1}(x)$, $e_a^i(x) = \lambda^{-1}(x)u_a^i(x)$, but one must bear in mind that this intensity is not of an individual spin, because it collects correlations contained in the microscopic wave function $|0\rangle$. (The limit $\lambda \rightarrow 0$ corresponds to an intense spin correlation).

The construction of the local momentum and spin operators inescapably leads to two fundamental distinct views of space, hereafter called M (material) and OP (order parameter) spaces. We have seen that remarkably the OP space, like M space, also has commuting momentum and spin operators, and so, quasiparticles over the correlated spin-charged background can be labeled by them. The price to pay to have such operators is that the OP space becomes a curved space. The M space is that of the atoms, periodically arranged, where spins and charges hop from on site to the other. Such view is depicted in Fig.(2). For instance, the Hubbard model provides a view of the Mott insulator in M space. The OP space is that of the macroscopic superconducting state, which is intrinsically delocalized, and does not feel the individual sites, but the average correlation defined by Eqs.(15) (quantum view) and (16) (classical view). We introduce a notation to distinguish indices associated to M and OP spaces, although both are three-dimensional: $a, b, c, d = 1, 2, 3$ and $i, k, j, l = 1, 2, 3$ correspond to M and OP spaces, respectively. Notice that an inverse transformation can be defined by $\sigma_j(x) = e_j^b(x)\sigma_b$, and because

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$\sigma^j \sigma_j = \sigma^b \sigma_b$, we have the general properties $e_a^i(x) e_j^a(x) = \delta_j^i$, $e_j^b(x) e_a^j(x) = \delta_a^b$, where δ always refers to the Kronecker delta (identity matrix). There is no need to distinguish between upper and lower indices in M space ($\sigma^a = \sigma_a$) and throughout this paper our choice between them is purely done on aesthetical grounds. But in OP space the situation is different, one must distinguish between them, ($\sigma^i \neq \sigma_i$), as upper and lower indices describe the transformed set and its inverse, respectively. We shall also refer to the upper (lower) indices as contravariant (covariant) indices.

For later purposes we briefly review well known features of the momentum and spin operators of an independent particle. (1a) The position space representation of the momentum operator is $P_a \equiv (\hbar/i) \partial_a$. The spin operator is described by the Pauli matrices, σ_a , whose anti-commutator and the commutator relations are given by $\{\sigma_a, \sigma_b\} = 2\delta_{ab}$, and $[\sigma_a, \sigma_b] = 2i\epsilon_{abc} \sigma_c$, respectively. (2a) The momentum and the spin operator components commute according to Eq.(13). (3a) In real space the momentum operator satisfies the product (Leibniz) rule, $P_a(\phi_{index} \chi_{index'}) = (P_a \phi_{index}) \chi_{index'} + \phi_{index} (P_a \chi_{index'})$. (4a) The momentum operator components are commutative, $[P_a, P_b] = 0$. The tensors ϕ_{index} and $\chi_{index'}$ contain sets of indices, *index* and *index'*, with an arbitrary combinations of spatial ($a, b, c, d = 1, 2, 3$) and spinorial ($\alpha, \beta = 1, 2$) indices. Consistency with (1a) and (2a) implies that the action of the momentum operator on some special tensors gives zero, namely, the Kronecker delta, $P_a \delta_{bc} = 0$, and the totally anti-symmetric tensor, $P_a \epsilon^{bcd} = 0$. This last tensor takes values $\epsilon^{bcd} = 1, -1, 0$, for cyclic, anti-cyclic and repeated indices, respectively.

Next we explore the consequences of the local spin operator defined in OP space, due to Eq.(16). The anti-commutation and the commutation of the Pauli matrices become,

$$\{\sigma^i(x), \sigma^j(x)\} = 2g^{ij}(x), \quad (17)$$

$$g^{ij}(x) = e_a^i(x) e_a^j(x), \quad (18)$$

$$[\sigma^i(x), \sigma^j(x)] = 2i\epsilon^{ijk}(x) \sigma_k(x), \quad (19)$$

$$\epsilon^{ijk}(x) = e^{-1} \epsilon^{ijk}. \quad (20)$$

The determinant of the spin-frame satisfies the condition

$$e_a^i e_b^j e_c^k \epsilon^{abc} = e^{-1} \epsilon^{ijk}. \quad (21)$$

Notice that Eq.(19) contains both the transformation and its inverse, the left side of it the transformed spins, whereas its right the inverse of this transformation. The determinant of e_a^i is denoted by e^{-1} , and is given by Eq.(21), which contains the totally anti-symmetric tensors ϵ^{abc} and ϵ^{ijk} , taking values 1,-1 and 0. However ϵ^{abc} is a tensor in M space but ϵ^{ijk} is not a tensor in OP space. The antisymmetric tensor in OP, ϵ^{ijk} , is defined in Eq.(20). Thus there are the two distinct notations, ϵ and ϵ , for the antisymmetric tensors in M and OP space, respectively.

Next we summarize the properties of local momentum and spin operators in OP space. (1b) There is a momentum operator, ∇_i , also called the covariant derivative.

The spin operator, $\sigma^i(x)$, has anti-commutator and commutator relations given by Eqs.(17) and (19), respectively. (2b) The momentum and the spin operator components commute according to Eq.(14). (3b) The momentum operator satisfies the product (Leibniz) rule, $\nabla_i(\phi_{index}\chi_{index'}) = (\nabla_i\phi_{index})\chi_{index'} + \phi_{index}(\nabla_i\chi_{index'})$. (4b) The momentum operator components do not commute, $[\nabla_i, \nabla_j] \neq 0$ because of the spin-charged background. The tensors ϕ_{index} and $\chi_{index'}$ contain sets of indices, $index$ and $index'$, with an arbitrary combinations of M space ($a, b, c, d = 1, 2, 3$), OP space ($i, j, k, l = 1, 2, 3$), and spinorial ($\alpha, \beta = 1, 2$) indices.

Similarly to the free independent particle case, the covariant derivative applied to the special tensors must also vanish by consistency or by use of its explicit form²¹. The covariant derivative commutes with both M and OP space spin operators, namely $[\nabla_i, \sigma_a] = 0$ and Eq.(14) holds, and from this one obtains that,

$$\nabla_i \sigma_a = 0 \quad \text{and} \quad (22)$$

$$\nabla_i \sigma^j(x) = 0, \quad (23)$$

this last equation being another way to express the Fock-Ivanenko condition⁵² given in Eq.(14). From the anticommutator and the commutator relations, given by Eqs.(17) and (19), respectively, it follows that the tensors defined by Eqs.(18) and (20) satisfy,

$$\nabla_i g^{jk}(x) = 0 \quad \text{and} \quad (24)$$

$$\nabla_i \varepsilon^{jkl}(x) = 0. \quad (25)$$

The Kronecker delta in M and OP space vanish under the covariant derivative, $\nabla_k(\delta_m^j) = 0$, $\nabla_k(\delta^{ab}) = 0$, and also does the totally antisymmetric symbol in M space, $\nabla_k(\epsilon^{abc}) = 0$. The lack of commutativity between momentum components introduces novel physical aspects to this theory brought by the spin-charged background. We recall that in M space this commutativity is also lost by the presence of a magnetic field. According to Eq.(18), and the Leibniz rule, the covariant derivative applied to the spin-frame must vanish,

$$\nabla_k e_a^i(x) = 0. \quad (26)$$

The vanishing of the covariant derivative applied to all special tensor is a straightforward consequence of the above condition, which is then the most important of all relations. This is the condition introduced by Cartan which is in an equal footing to Eq.(14). Notice that all the above relations follow from consistency without making use of the explicit form of this covariant derivative, which is only done in the next section. In its explicit form the above equation is given by,

$$\nabla_k e_a^i = \frac{\hbar}{i} [\partial_k e_a^i - g\omega_{kab} e_b^i + \Gamma_{km}^i e_a^m], \quad (27)$$

A detailed discussion of the above expression is carried elsewhere²¹. Notice that the covariant derivative contains two new sets of fields: (1) *the spin connection*, which

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carries OP and M space indices, ω_{iab} , and is antisymmetric in ab ($\omega_{iba} = -\omega_{iab}$); (2) the affine connection with just OP indices, Γ_{im}^k .

The metric tensor g^{ij} that naturally arises in Eq.(18) describes the infinitesimal distance between nearest points in OP space separated by dx_i . To see this just consider the internal product $\sigma^i dx_i = \sigma^a dx_a$, and take the anticommutator $\{\sigma^i dx_i, \sigma^j dx_j\}/2 = g^{ij} dx_i dx_j = \{\sigma^a dx_a, \sigma^b dx_b\}/2 = dx_a dx_a$ since in M space the metric is the Kronecker delta. Therefore we reach the conclusion that OP is curved and the concept of distance must be defined as $ds^2 = g^{ij} dx_i dx_j$. We stress that uncorrelated spins do not curve the OP space. We have argued before that if all spins are independent rotations of the reference spin, then the spin-frame is a pure rotation, $e_b^i(x) = u_b^i(x)$ and this renders the metric, according to Eq.(18), trivial, $g^{ij} = \delta^{ij}$ since $u_a^i(x)u_a^j(x) = \delta^{ij}$. Two nearest points in OP and M spaces, are related by $dx^i = e_a^i(x) dx^a$, or equivalently, $e_a^i(x) = \partial x^i / \partial x^a$. From this it follows that under a general coordinate transformation,

$$e'^i{}_a(x') = \frac{\partial x'^i}{\partial x^a} = \frac{\partial x'^i}{\partial x^j} \frac{\partial x^j}{\partial x^a} \equiv \Lambda_j^i e_a^j(x). \quad (28)$$

Apply the determinant to both sides and consider that $e^{-1} \equiv \det(e_a^i)$, to obtain that $e'^{-1} = \det(\frac{\partial x'^i}{\partial x^j}) e^{-1}$. This means that a volume in OP space must be corrected according to $e' d^3 x' = e d^3 x$. The determinant, $1/e$, is a measure of the intensity of the spin correlation $e_a^i(x)$, and shows that regions of high spin correlation ($e \rightarrow 0$) have less effective volume to integrate in OP space than those of uncorrelated spins ($e \rightarrow 1$). Thus spin and superconductivity are indeed competing effects in this formulation too.

4. Gauge symmetry of local rotations

The derivation of the local momentum and spin operators set over the spin-charged background is a key element to obtain the kinetic energy of the condensate²¹. The symbol D_j used to express the covariant derivative applied to Ψ , as shown in Eq.(1), is given by,

$$D_j \Psi \equiv \left(\frac{\hbar}{i} \partial_j - \frac{\hbar g}{2} \omega_{jab} \Sigma^{ab} - \frac{q}{c} A_j \right) \Psi. \quad (29)$$

Notice the presence of a new coupling constant g to describe the interaction with the spin-charged background. The hermitian matrices Σ_{ab} are the generators of the rotation group and satisfy the commutation rule, $[\Sigma_{ab}, \Sigma_{cd}] = i\delta_{ac}\Sigma_{bd} + i\delta_{bd}\Sigma_{ac} - i\delta_{ad}\Sigma_{bc} - i\delta_{bc}\Sigma_{ab}$, but for the two-component OP they become Pauli matrices, $\Sigma_{ab} = -i[\sigma_a, \sigma_b]/2 = \epsilon_{abc}\sigma^c$. Therefore besides the spin-frame $e_a^i(x)$, the interaction of the OP with the background also demands the spin connection $\omega_{jab}(x)$. A scalar OP, such as in case of the standard Ginzburg-Landau theory, can not minimally couple to the spin-charged background, because the covariant derivative is simply given by $\nabla_i \psi = (\hbar/i)\partial_i \psi$. Therefore the superconducting OP must have at least two components to develop minimal coupling to a spin-charged background.

Thus minimal coupling introduces in the kinetic energy a $SU_L(2) \otimes U_L(1)$ symmetry, where L stands for local. Under a local rotation $U(x) = \exp(i\theta_{ab}\Sigma^{ab})$ the OP transforms as $\Psi' = U\Psi$ and invariance under local rotation, $\nabla_j(\Psi') = U\nabla'_j\Psi$, sets the way the spin connection must transform, $\omega_j'^{ab}\Sigma_{ab} = U^{-1}\omega_j^{ab}\Sigma_{ab}U - (2/i g)U^{-1}\partial_j U$. The spin connection plays the role of the vector potential in the non-abelian gauge (Yang-Mills) theory^{27,38}. Under a $U_L(1)$ rotation, $\Psi' = \exp(i\theta)\Psi$, the vector potential transforms, as below, to have the gauge invariance of electromagnetism: $A'_j = A_j - (c\hbar/q)\partial_j\theta$. The commutator becomes,

$$[D_i, D_j] = -\frac{\hbar q}{ic}F_{ij} - \frac{\hbar^2 g}{2i}R_{ij}^{ab}\Sigma_{ab}, \quad (30)$$

$$F_{ij} = \partial_i A_j - \partial_j A_i, \quad (31)$$

$$R_{ij}^{ab}(\omega) = \partial_i \omega_j^{ab} - \partial_j \omega_i^{ab} - g(\omega_i^{ac}\omega_j^{bc} - \omega_i^{bc}\omega_j^{ac}). \quad (32)$$

The spin-charged background modifies the commutativity of the momentum components and besides the electromagnetic field F_{ij} , there is also the Riemannian curvature tensor $R_{ij}^{ab}(\omega)$. Using Eq.(27) the second derivative of the spin frame becomes $(\partial_l \partial_k - \partial_k \partial_l)e_a^i = gR_{lk}^{ab}(\omega)e_b^i - R_{lkj}^i(\Gamma)e_a^j$. We impose the condition that the spin-frame has a smooth spatial behavior, such that $(\partial_l \partial_k - \partial_k \partial_l)e_a^i = 0$, to obtain that, $gR_{lk}^{ab}(\omega)e_b^i e_a^j = R_{lkj}^i(\Gamma)$. This means that the Riemannian curvature tensor, already expressed in terms of the spin connection in Eq.(32), can also be written solely in terms of the affine connection: $R_{lkj}^i(\Gamma) = \partial_l \Gamma_{kj}^i - \partial_k \Gamma_{lj}^i - (\Gamma_{km}^i \Gamma_{lj}^m - \Gamma_{lm}^i \Gamma_{kj}^m)$. In Kleinert's description of defects, the rotational symmetry is absent, and so, ω_i^{ab} is zero. Then the curvature necessarily has singular behavior, and so does the spin-frame field, since in this case one obtains that $R_{lkj}^i(\Gamma) = e_j^a (\partial_l \partial_k - \partial_k \partial_l)e_a^i$. This shows that the Kleinert's limit is not useful for the purpose of describing the non-singular spin-charged background treated here.

5. The $SU_L(2) \otimes U_L(1)$ invariant ground state condition

For the two-component OP the kinetic energy can be expressed in two different but equivalent ways given below:

$$F_k = \frac{1}{2m} \int \frac{d^3x}{V} e \{ g^{ij} (D_i \Psi)^\dagger (D_j \Psi) - \hbar^2 g R \Psi^\dagger \Psi - \frac{\hbar}{2} \Gamma_{lm}^j \varepsilon^{lmk} [\Psi^\dagger \sigma_k (D_j \Psi) + (D_j \Psi)^\dagger \sigma_k \Psi] \}, \quad \text{and}, \quad (33)$$

$$F_k = \frac{1}{2m} \int \frac{d^3x}{V} e \{ (\sigma^i D_i \Psi)^\dagger (\sigma^j D_j \Psi) + \frac{\hbar q}{c} h^k \Psi^\dagger \sigma_k \Psi \}. \quad (34)$$

The equality between Eqs.(33) and (34) is the Weitzenböck-Liebnorowitz identity⁴⁹. The derivation of this identity will be given elsewhere²¹. We stress that such derivation relies on the commutativity between local spin and momentum operators, this last one being essentially the covariant derivative. The most natural way to express the kinetic energy is through Eq.(33), because of the gradient

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square term, $(D_i \Psi)^\dagger (D_j \Psi)$. Notice the presence of new and important contributions brought by the spin-charged background, namely, the scalar Riemannian curvature R , and the torsion field $\Gamma_{lm}^j \varepsilon^{lmk}$. The local magnetic field and the Riemannian scalar curvature are given by,

$$h^k \equiv \frac{1}{2} \varepsilon^{ijk} F_{ij} \quad \text{and} \quad (35)$$

$$R \equiv e_a^i e_b^j R_{ij}^{ab}, \quad (36)$$

respectively. Eq.(33) has a temperature like term, $gR\Psi^\dagger\Psi$, in the kinetic energy of the condensate, which embodies our description of inhomogeneities by the present formalism. It is well-known that in the Ginzburg-Landau theory a term proportional to $|\Psi|^2$ sustains superconductivity because it flips sign in the critical temperature, being negative below and positive above. The Riemannian scalar curvature R also changes sign from one region to another, depending on the spin-charged background, leading to a naturally heterogeneous order parameter, and so, to inhomogeneous superconductivity.

The electromagnetic current follows from the linear term of the kinetic energy, proportional to the vector potential,

$$F_k = \int \frac{d^3x}{V} e \left(-\frac{1}{c} J^i A_i + \dots \right), \quad (37)$$

and consequently there are also two equivalent expressions for it:

$$J^i = \frac{q}{2m} g^{ij} [\Psi^\dagger (D_j \Psi) + (D_j \Psi)^\dagger \Psi] + \frac{\hbar q}{2m} \Gamma_{kl}^i \varepsilon^{klm} \Psi^\dagger \sigma_m \Psi, \quad \text{and} \quad (38)$$

$$J^i = \frac{q}{2m} [(\sigma^j D_j \Psi)^\dagger \sigma^i \Psi + \Psi^\dagger \sigma^i (\sigma^j D_j \Psi)] - \frac{\hbar q}{2m} \varepsilon^{ijk} \partial_j (\Psi^\dagger \sigma_k \Psi), \quad (39)$$

At this point it becomes clear how the background interacts electromagnetically with the superconducting OP. The kinetic energy and the supercurrent expressions, given by Eq.(33) and Eq.(38), respectively, show that the torsion field, $\Gamma_{kl}^i \varepsilon^{klm}$, known to describe dislocations in a crystal^{28,33,32}, contributes electromagnetically to the supercurrent according to Eq.(38). To obtain Ampère's law firstly take the field energy,

$$F_f = \int \frac{d^3x}{V} e \frac{h^k h_k}{8\pi}, \quad (40)$$

and then the current contribution to the kinetic energy, to obtain that,

$$\varepsilon^{ijk} \partial_k h_i = \frac{4\pi}{c} J^j. \quad (41)$$

The contravariant magnetic field h^k defines the *magnetic induction* B^k , because the volume integral turns into a line integral by means of $\varepsilon^{klm} = \epsilon^{klm}/e$. The magnetic

induction is a topological number that counts the number of vortices, and so, is proportional to an integer vector $\vec{N} = (N_1, N_2, N_3)$. For a cubic unit cell with size $V^{1/3}$, with periodic boundary conditions²⁰, one obtains that $B^k \equiv \int d^3x e h^k / V = \int d^3x \epsilon^{klm} \partial_l A_m / V = 2\pi \Phi_0 N^k / V^{2/3}$. Thus we reach the conclusion that ground state equations solve exactly Ampère's law, Eq.(41), thanks to the current, Eq.(39) because the first ground state equation, Eq.(11) leads to the second one, Eq.(12).

Gauge field theories have been proposed to describe disordered systems^{58,59,60,54} because a disordered system, in its most general form, breaks the translational invariance in every point of space, but locally preserves the rotational invariance. Therefore at each point one defines a new set of axis, and so, there is local rotational invariance which becomes a non-abelian gauge symmetry with the rotation group playing the role of the internal symmetry⁵⁴. Essentially we have extended this proposal to describe disordered systems to the superconducting state in presence of a spin-charged background.

6. Conclusion

The present approach describes how the order parameter feels an inhomogeneous background. For the one-component OP, ψ , such inhomogeneous background can be thought as a local distribution of critical temperatures, $T_c(x)$, that turns the condensate energy, $\alpha_0(T - T_c(x))|\psi|^2 + \beta|\psi|^4/2$, local. Some regions will have the temperature above the local critical temperature, $T > T_c(x)$, whereas others not, $T < T_c(x)$. The condensate energy can be locally positive or negative, suggesting a possible coexistence of normal ($\psi = 0$) and superconducting regions ($\psi \neq 0$). However a full response to the size of these regions requires that we take into account the kinetic energy, which connects all points into a single state. Thus one must add the kinetic to condensate energy to find the energetic cost of interfaces separating the normal to the superconducting regions. For the present case description of a two-component OP, Ψ , we have obtained in this paper a new mechanism to locally sustain superconductivity based only on the kinetic energy. This mechanism relies on the fact that the kinetic energy contains a term $R|\Psi|^2$, where the Riemannian spatial curvature R is induced by spin correlations and charge dislocations. This curvature can be either positive or negative, and similarly to a local critical temperature turns $-R|\Psi|^2$ into a locally negative or positive term. This term added to the traditional gradient square term present in the kinetic energy, as seen in Eq.(33), sustains a spatially corrugated OP.

In summary we propose here an approach to describe a superconducting layer through a two-component order parameter interlaced with extra spin and charge degrees of freedom. The order parameter and the local magnetic field are determined by first order differential equations in presence of the spin-charged background. The obtainment of local momentum and spin operators, constructed over the spin-charged background, leads to a twofold view of the kinetic energy because of the Weitzenböck-Liebnorowitz identity. This also leads to a twofold formulation of the

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supercurrent, and to the solution of Ampère's law, yielding the sought first order differential equations. The local commuting spin and momentum operators show that the order parameter lives in a curved space with torsion, as described by the geometrical approach of Élie Cartan. The ground state displays a non-abelian gauge symmetry set by local invariance under rotations. The ground state equations are meant to describe the order parameter ranging from the strong spin correlation regime to the charged dislocation regime.

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